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New Parameterization in Muon Decay and the Type of Emitted Neutrino. II

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In a previous paper, new sets of parameters to replace the Michel parameters were proposed to analyze data for the muon decay $\mu^+ \to e^+\nu_e\overline{\nu_\mu}$. Both $(V-A)$ and $(V+A)$ charged currents with finite neutrino mass have been used to study this decay. In the present paper, this parameterization is extended to a more general form, and a method for data analysis (least squares) is discussed for the propose of determining the rate of contribution from the $(V+A)$ current. We find that there is a simple form in which the set of parameters is related primitively to the physical quantities. It is shown that the Michel parameters are one of the other sets that are obtained from this simple form by rearranging one term. We derive the condition to obtain the same information regarding unknown physical quantities in the case that the data are analyzed using these simple and rearranged forms separately. We find that there is some possibility to get different results from these analyses, because the equivalent condition is very delicate and the QED radiative corrections should be treated carefully. We propose a consistent formula for data analysis. It is useful to compare the value obtained in the least squares fit using the simple form with that obtained using the prediction of the standard model, because a large difference is not expected, especially in the case of the Majorana neutrino. Finally, we point out that the method we proposed to determine the type of neutrino in the previous paper is incorrect.

§1. Introduction

Normal muon decay has been studied as a tool with high statistics to determine the structure of the weak interaction. The purpose of this paper is to investigate the effect of the $(V+A)$ current added to the standard model and to find a consistent formula to treat the QED corrections in the data analysis on the basis of the method of least squares. Both the Dirac and Majorana neutrino cases are examined.

Recently, the TWIST group[1)] reported precise experimental data and analyzed them using the helicity preserving four fermion weak interaction with $(S\pm P)$, $(V\pm A)$ and T forms for the case of massless neutrino.[2)] They used the expression on the basis of the Michel parameterization for the e^+ energy spectrum

$$\frac{d^2\Gamma}{dx\,d\cos\theta} \propto \Big[\mathcal{N}(x) + P_\mu \cos\theta\,\mathcal{P}(x)\Big], \tag{1·1}$$

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where

$$\mathcal{N}(x) = 6\,x^2\left[(1-x) + \frac{2}{9}\,(4x-3)\,\rho_M\right], \tag{1·2}$$

$$\mathcal{P}(x) = 2\,x^2\,\xi_M\left[(1-x) + \frac{2}{3}\,(4x-3)\,\delta_M\right]. \tag{1·3}$$

Here, x is defined as $x \equiv E/W$, where E is the energy of the emitted positron and $W = (m_\mu^2 + m_e^2)/(2\,m_\mu)$, and m_μ and m_e are the muon and positron masses, respectively. The angle θ is the direction of the emitted e^+ with respect to the muon polarization vector $\vec{P_\mu}$ at the instant of the μ^+ decay. In the above expressions, to simplify the following explanation, we do not include terms proportional to m_e or neutrino masses, and also, QED radiative corrections are omitted.

The standard model predicts $\rho_M = \delta_M = 0.75$ and $\xi_M = 1$ for these Michel parameters. The conventional method of experimental data analysis has been to determine the deviations from these predicted values. The new experimental values reported by the TWIST group[1)] are[*)]

$$\rho_M = 0.75080 \pm 0.00032(\mathrm{stat}) \pm 0.00097(\mathrm{syst}) \pm 0.00023, \tag{1·4}$$

$$\delta_M = 0.74964 \pm 0.00066(\mathrm{stat}) \pm 0.00112(\mathrm{syst}), \tag{1·5}$$

$$0.9960 < P_\mu\xi \leqq \xi < 1.0040 \quad (90\%\mathrm{CL}). \tag{1·6}$$

Note that the QED radiative corrections are taken into account in their analysis. Because these deviations are small, it is useful to determine them directly.

In a previous paper,[3)] which is referred to as " I " hereafter, we proposed using new parameters suitable for investigating these deviations. We showed that various parameterizations are allowed by making different choices of the normalization factor for the isotropic part of the energy spectrum. Among them, we mainly discussed the specific one directly related to the Michel parameters. We refer to this as the Michel parameterization. We assumed that the weak interaction Hamiltonian consists of both $(V-A)$ and $(V+A)$ charged currents and that the neutrino has finite mass. As a result, we have two kinds of lepton mixing matrices and three coupling constants which represent the rates of mixture of the $(V+A)$ current. We refer to these unknown quantities as the "weak mixing constants".

In this paper, a more general parameterization is investigated within the framework of the same Hamiltonian. We find a set of parameters expressed in a simple form in terms of a combination of the weak mixing constants. We show that the specific form corresponding to the Michel parameterization is one of many forms that are derived from this simple form by rearranging a term in it. Sets of parameters realized through such rearrangement are related to combinations of the weak mixing constants in somewhat complicated manners. Of course, these different sets

[*)] The last uncertainty of ρ_M represents its dependence on an additional Michel parameter η_M. The term associated with η_M is not included in Eq. (1·2), because its contribution is proportional to m_e in general, as discussed in §2. The η_M parameter itself becomes to be zero in some model. If it is the case, the last uncertainty of ρ_M should be set to be zero. The TWIST group assumed $\eta_M = -0.007 \pm 0.013$.

of parameters should contain the same information with regard to the weak mixing constants. In §3, we seek a condition to obtain the same information with different parameterizations in the case that experimental data are analyzed using the method of least squares. It is pointed out that this condition is very delicate and we should therefore be careful in treating QED radiative corrections in the data analysis.

In I, we also proposed a method for discriminating between the Dirac and Majorana neutrino experimentally using the method of least squares for the e^+ energy spectrum. However, we show in §2 of the present paper that this method is incorrect. Making such a discrimination is not easy in the case of muon decay. This point is discussed in §4.

In §2, we present thorough discussion of the general form of parameterization. Then, the present experimental bounds for the new parameters are listed. In §3, we study a condition for obtaining the information concerning the unknown weak mixing constants and a method for appropriately taking the QED radiative corrections into account. We propose a consistent formula for the method of least squares in the data analysis. In §4, we give some comments. Appendix A contains expressions for the polarization of the emitted positron.

§2. Parameterization of the decay spectrum

We assume the following form of the effective weak interaction Hamiltonian for μ^+ decay:[4)]

$$\mathcal{H}_W(x) = \frac{G_F}{\sqrt{2}} \left\{ j^\dagger_{eL\,\alpha} j^\alpha_{\mu L} + \lambda j^\dagger_{eR\,\alpha} j^\alpha_{\mu R} + \eta j^\dagger_{eR\,\alpha} j^\alpha_{\mu L} + \kappa j^\dagger_{eL\,\alpha} j^\alpha_{\mu R} \right\} + \text{H.c.} \tag{2·1}$$

Here, G_F is the Fermi coupling constant.[5)] The coupling constants (λ, η and κ) represent the rates of mixture of the $(V+A)$ current for the combination of the left(right)-handed charged leptonic currents $j_{\ell L(R)}$.[*)] These currents are defined as

$$j_{\ell L\,\alpha}(x) = \sum_{j=1}^{2n} \overline{E_\ell(x)} \gamma_\alpha (1-\gamma_5) U_{\ell j} N_j(x), \tag{2·2}$$

$$j_{\ell R\,\alpha}(x) = \sum_{j=1}^{2n} \overline{E_\ell(x)} \gamma_\alpha (1+\gamma_5) V_{\ell j} N_j(x) \tag{2·3}$$

for the case of n generations. Here, $U_{\ell j}$ and $V_{\ell j}$ are the left- and right-handed lepton mixing matrices, and E_ℓ and N_j represent, respectively, the mass eigenstates

*) In order to understand the physical meaning of the coupling constants, let us consider a typical example in gauge theory, the $SU(2)_L \times SU(2)_R \times U(1)$ model with left- and right-handed weak gauge bosons, W_L and W_R. The coupling constants in Eq. (2·1) are related to physical quantities as

$$\lambda \sim (\lambda_c + \tan^2\zeta), \qquad \kappa = \eta \sim (-\tan\zeta).$$

Here, we have $\lambda_c = (M_1/M_2)^2$, where M_1 and M_2 are the masses of the mass-eigenstate gauge bosons, which are expressed in terms of the weak eigenstate gauge bosons W_L and W_R with the mixing angle ζ. (For example, see Appendix A of I.)

of charged leptons and neutrinos. Throughout this paper, neutrinos are assumed to have finite masses, $m_\nu \neq 0$.

The decay spectrum of e^+ in the rest frame of the polarized μ^+ is defined as

$$\frac{d^2\Gamma}{dx\, d\cos\theta} = \Gamma_W\, A\left[\mathcal{N}(x) + P_\mu\, \cos\theta\, \mathcal{P}(x)\right], \tag{2·4}$$

where the sum over the spin of e^+ has been taken and we have

$$\Gamma_W = \frac{m_\mu\, G_F^2\, W^4}{12\,\pi^3}\,. \tag{2·5}$$

The isotropic and anisotropic parts of the energy spectrum obtained from the leptonic Hamiltonian in Eq. (2·1) are expressed as[3),6)]

$$\mathcal{N}(x) = \left(\frac{1}{A}\right)\left[a_+\, n_1(x) + (\, k_{+\,c} + \varepsilon_m\, k_{+\,m}\,)\, n_2(x) + \varepsilon_m\, \lambda\, d_r\, n_3(x)\right], \tag{2·6}$$

$$\mathcal{P}(x) = \left(\frac{1}{A}\right)\left[a_-\, p_1(x) + (\, k_{-\,c} + \varepsilon_m k_{-\,m}\,)\, p_2(x)\right], \tag{2·7}$$

where the decay formulae for the Dirac and Majorana neutrinos are obtained by setting $\varepsilon_m = 0$ and $\varepsilon_m = 1$, respectively.

The constant A is introduced to Eqs. (2·4), (2·6) and (2·7) to simplify the coefficient of the prediction obtained from the standard model in $\mathcal{N}(x)$. Its explicit form is given below [see Eq. (2·23)]. This A is called a "normalization factor", following I.

The x dependent parts of $\mathcal{N}(x)$ and $\mathcal{P}(x)$ are defined as follows:*)

$$n_1(x) = x_p\,(3x - 2x^2 - x_0^2), \tag{2·8}$$

$$n_2(x) = 12\, x_p\, x\,(1 - x), \tag{2·9}$$

$$n_3(x) = 6\, x_p\, x_0\,(1 - x), \tag{2·10}$$

$$p_1(x) = x_p^2\,(-1 + 2\,x - r_0^2), \tag{2·11}$$

$$p_2(x) = 12\, x_p^2\,(1 - x), \tag{2·12}$$

where

$$x_p = \sqrt{x^2 - x_0^2}, \quad x_0 = \frac{m_e}{W} = 9.7\cdot 10^{-3}, \quad r_0^2 = \frac{m_e^2}{m_\mu\, W} = 4.7\cdot 10^{-5}. \tag{2·13}$$

The first terms, $n_1(x)$ and $p_1(x)$, in $\mathcal{N}(x)$ and $\mathcal{P}(x)$, respectively, are the predictions of the standard model. They are called the "standard functions" in this

*) The spectrum functions $\mathcal{N}(x)$ and $\mathcal{P}(x)$ in Eqs. (2·6) and (2·7) are precise, except in a very small range from the maximum of x, $x_{\max} = 1 - [(m_j + m_k)^2/(2\, m_\mu\, W)]$. Here, m_j and m_k are masses of two emitted neutrinos. This is because two kinds of kinematical factors come from the phase space integral over the emitted neutrinos, and they can be set to unity over almost the entire range of x. In other words, they possess significant x dependence only in an extremely narrow range of order $(m_\nu/m_\mu)^2 < O(10^{-16})$ near $x_{\max}$. It should be noted that $\mathcal{N}(x)$ and $\mathcal{P}(x)$ become zero sharply at $x_{\max}$ due to these kinematical factors. (For details, see §2.1 of I.)

paper. The others, $n_2(x)$, $n_3(x)$ and $p_2(x)$, are called the "deviation functions". In these functions, all terms proportional to the neutrino mass are ignored, because of the smallness of (m_ν/m_μ) which is less than $9 \cdot 10^{-9}$. Here, m_ν represents a typical mass scale of emitted neutrinos and is assumed to satisfy $m_\nu < 1\mathrm{eV}$.

The coefficients $a_\pm$, $k_{\pm\,c}$, $k_{\pm\,m}$ and d_r in $\mathcal{N}(x)$ and $\mathcal{P}(x)$ are constants. They are obtained by summing some products of the coupling constants (λ, η and κ) and the lepton mixing matrices ($U_{\ell\,j}$ and $V_{\ell\,j}$) only over the emitted neutrinos. In the Dirac neutrino case, they are defined by

$$a_\pm = \left(1 \pm \lambda^2\right), \qquad k_{\pm\,c} = \left(\frac{1}{2}\right)\left(\kappa^2 \pm \eta^2\right). \tag{2·14}$$

Here, it is assumed that all neutrinos can be emitted in the μ decay, and we have used the relation

$$\Sigma_j |U_{\ell j}|^2 = \Sigma_j |V_{\ell j}|^2 = 1, \tag{2·15}$$

coming from the unitarity condition, because j in the sum runs over all n neutrinos.

By contrast, in the Majorana neutrino case, we assume that there exist n additional heavy neutrinos that are not emitted in the decay. These coefficients are given as follows:

$$a_\pm = \left[\left(1 - \overline{u_e}^{\,2}\right)\left(1 - \overline{u_\mu}^{\,2}\right) \pm \lambda^2\, \overline{v_e}^{\,2}\, \overline{v_\mu}^{\,2}\right], \tag{2·16}$$

$$k_{\pm\,c} = \left(\frac{1}{2}\right)\left[\kappa^2\left(1 - \overline{u_e}^{\,2}\right)\overline{v_\mu}^{\,2} \pm \eta^2\, \overline{v_e}^{\,2}\left(1 - \overline{u_\mu}^{\,2}\right)\right], \tag{2·17}$$

$$k_{\pm\,m} = \left(\frac{1}{2}\right)\left[\,\kappa^2\,|\,\overline{w_{e\mu}}\,|^2 \pm \eta^2\,|\,\overline{w_{e\mu\,h}}\,|^2\,\right], \tag{2·18}$$

$$d_r = \left(\frac{1}{2}\right)\mathrm{Re}(\overline{w_{e\mu}}^{\,*}\ \overline{w_{e\mu\,h}}). \tag{2·19}$$

Here, $\overline{u_\ell}^{\,2}$, $\overline{v_\ell}^{\,2}$, $\overline{w_{e\mu}}$ and $\overline{w_{e\mu h}}$ are all small quantities which represent the extent of the deviation from the unitarity condition due to the existence of heavy neutrinos:

$$\Sigma_j' |U_{\ell j}|^2 \equiv 1 - \overline{u_\ell}^{\,2}, \qquad \Sigma_j' |V_{\ell j}|^2 \equiv \overline{v_\ell}^{\,2}, \tag{2·20}$$

$$\Sigma_j'\, U_{ej}\, V_{\mu j} \equiv \overline{w_{e\mu}}, \qquad \Sigma_k'\, V_{ek}\, U_{\mu k} \equiv \overline{w_{e\mu\,h}}, \tag{2·21}$$

where the primed sum is taken over only the n light neutrinos of the $2n$ total neutrinos. Their orders of magnitudes are $\overline{u_\ell}^{\,2} \sim \overline{v_\ell}^{\,2} \sim O((m_{\nu D}/m_{\nu R})^2)$ and $\overline{w_{e\mu}} \sim \overline{w_{e\mu h}} \sim O(m_{\nu D}/m_{\nu R})$, if the seesaw mechanism is assumed.[7),*)] Here, $m_{\nu D}$ and $m_{\nu R}$ are, respectively, representatives of the Dirac-type and right-handed Majorana-type masses in the neutrino mass matrix.

2.1. *Isotropic part of the spectrum: $\mathcal{N}(x)$*

Let us first consider the isotropic part, $\mathcal{N}(x)$. For the purpose of surveying the deviation from the standard model, it is suitable to examine $\mathcal{N}(x)$ by treating the

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*) For details, see Appendix A and §2.2 of I as an example.

standard function $n_1(x)$ as the basis of analysis. In order to determine the relation to the Michel parameter introduced in Eq. (1·2), we rearrange $n_1(x)$ in $\mathcal{N}(x)$ as follows:

$$\mathcal{N}(x) = \frac{1}{A}\Big\{ n_1(x)[a_+ + s\,(k_{+\,c} + \varepsilon_m\, k_{+\,m}) + t\,\varepsilon_m\,\lambda\, d_r] \\ + [n_2(x) - s\,n_1(x)]\,(k_{+\,c} + \varepsilon_m\, k_{+\,m}) + [n_3(x) - t\,n_1(x)]\,\varepsilon_m\,\lambda\, d_r \Big\}, \quad (2·22)$$

where s and t are some arbitrary numbers. Here, the normalization factor A is set equal to the following $\mathcal{A}_{s\,t}$ in order to simplify the coefficient of $n_1(x)$:

$$\mathcal{A}_{s\,t} = a_+ + s\,(k_{+\,c} + \varepsilon_m\, k_{+\,m}) + t\,\varepsilon_m\,\lambda\, d_r > 0. \quad (2·23)$$

It is natural to restrict s and t to values satisfying the condition $\mathcal{A}_{s\,t} > 0$.

Thus, the isotropic part, denoted by $\mathcal{N}_{s\,t}(x)$, takes the form

$$\mathcal{N}_{s\,t}(x) = n_1(x) + [n_2(x) - s\,n_1(x)]\,\rho^{(s\,t)} + [n_3(x) - t\,n_1(x)]\,\eta^{(s\,t)}, \quad (2·24)$$

where the two parameters $\rho^{(s\,t)}$ and $\eta^{(s\,t)}$ are defined as

$$\rho^{(s\,t)} = \frac{k_{+\,c} + \varepsilon_m k_{+\,m}}{\mathcal{A}_{s\,t}} > 0, \qquad \eta^{(s\,t)} = \frac{\varepsilon_m \lambda\, d_r}{\mathcal{A}_{s\,t}}. \quad (2·25)$$

It is worthwhile to note that $\rho^{(s\,t)}$ is positive in the case of the Hamiltonian in Eq. (2·1), because its numerator is positive, as seen from Eqs. (2·14) and (2·16) – (2·18).

The two combinations $[n_2(x) - s\,n_1(x)]$ and $[n_3(x) - t\,n_1(x)]$ in Eq. (2·24) play roles of deviation functions for $\mathcal{N}_{s\,t}(x)$. In I, there appear some misleading statements. Specifically, the deviation functions presented there are not linearly independent, and thus all of their coefficients cannot be determined independently.*)

The simplest choice of s and t is $(s,\, t) = (0,\, 0)$. In this case, the isotropic part is expressed as follows:

$$\mathcal{N}_{0\,0}(x) = n_1(x) + n_2(x)\,\rho^{(0\,0)} + n_3(x)\,\eta^{(0\,0)}. \quad (2·26)$$

This is merely the original expression given in Eq. (2·6), with $A = \mathcal{A}_{0\,0} = a_+$. It should be noted that we have $A \neq 1$ in principle for the Hamiltonian employed here. Explicitly, if the right-handed charged weak current or the existence of a heavy Majorana neutrino is assumed, we have the following expressions from Eqs. (2·14) and (2·16), respectively:

$$\mathcal{A}_{0\,0} = \left(1 + \lambda^2\right) > 1 \qquad \text{for the Dirac neutrino case,} \quad (2·27)$$

$$\mathcal{A}_{0\,0} \approx \left(1 - \overline{u_e}^{\,2} - \overline{u_\mu}^{\,2}\right) \lessapprox 1 \qquad \text{for the Majorana neutrino case.} \quad (2·28)$$

*) In I, a normalization factor is denoted by $A_{n\,\ell}$. A shortcut to reconstruct linearly independent deviation functions is simply to set $n = \ell$. The presentation there can be corrected by taking $A_{n\,\ell} \to A_{n\,n}$ and $\rho_m \to 0$. Then, $A_{n\,n}$ in I corresponds to $\mathcal{A}_{2n\,0}$ in the present paper.

The parameters $\rho^{(0\,0)}$ and $\eta^{(0\,0)}$ are related to the weak mixing constants. For the Dirac neutrino case, they are

$$\rho^{(0\,0)} \approx \frac{1}{2}\left(\kappa^2 + \eta^2\right) > 0, \tag{2·29}$$

$$\eta^{(0\,0)} = 0, \tag{2·30}$$

while for the Majorana neutrino case, they are

$$\rho^{(0\,0)} \approx \frac{1}{2}\left[\kappa^2\,\left(\overline{v_\mu}^{\,2} + |\,\overline{w_{e\mu}}\,|^2\right) + \eta^2\,\left(\overline{v_e}^{\,2} + |\,\overline{w_{e\mu\,h}}\,|^2\,\right)\right] > 0, \tag{2·31}$$

$$\eta^{(0\,0)} \approx \frac{1}{2}\,\lambda\,\mathrm{Re}(\overline{w_{e\mu}}^{\,*}\;\overline{w_{e\mu\,h}}). \tag{2·32}$$

In these expressions, only the lowest order terms are kept by choosing $\mathcal{A}_{0\,0} \approx 1$. Note that we can get no direct information on λ from the isotropic spectrum $\mathcal{N}_{s\,t}(x)$ in the Dirac neutrino case. By contrast, in the Majorana neutrino case, the parameter $\eta^{(0\,0)}$ is proportional to λ. However, the orders of magnitude of both $\rho^{(0\,0)}$ and $\eta^{(0\,0)}$ themselves seem to be very small, as seen from Eqs. (2·31) and (2·32).

Concerning the relation with the Michel parameter ρ_M, the relevant term in Eq. (1·2) can be reproduced from the spectrum $\mathcal{N}_{s\,t}(x)$ of Eq. (2·24) by setting $(s,\,t) = (2,\,0)$. Then, we have the following deviation function $[n_2(x) - 2\,n_1(x)]$ and its associated parameter $\rho^{(2\,0)}$:

$$n_2(x) - 2\,n_1(x) \,\simeq\, 2x^2\,(3 - 4\,x)\,, \qquad \rho^{(2\,0)} = -\frac{2}{3}\left(\rho_M - \frac{3}{4}\right). \tag{2·33}$$

It should be noted that the behavior of $[n_2(x) - 2\,n_1(x)]$ in $\mathcal{N}_{2\,0}(x)$ is different from that of $n_2(x) \simeq 12\,x^2\,(1-x)$ in $\mathcal{N}_{0\,0}(x)$. This is discussed at the end of §4.

The full expression for the Michel parameterization contains another combination $n_3(x)\,\eta_M$, which is omitted in Eq. (1·2), because $n_3(x)$ is small, due to the factor of x_0, as seen from Eqs. (2·10) and (2·13).[2)] There is the corresponding combination $n_3(x)\,\eta^{(2\,0)}$ in $\mathcal{N}_{2\,0}(x)$. Within the framework of the weak interaction Hamiltonian in Eq. (2·1), we have $\eta^{(20)} = 0$ in the Dirac neutrino case, while $\eta^{(20)} \neq 0$ in the Majorana neutrino case, as it is defined in Eq. (2·25). These two parameters η_M and $\eta^{(2\,0)}$ originate from different theoretical models of the weak interaction, as discussed in §4. But phenomenologically they yield the same experimental values. Below we list the present experimental restrictions which are derived from the averages summarized by the Particle Data Group:[2)]

$$\rho^{(2\,0)} = -(6 \pm 7)\cdot 10^{-4}, \tag{2·34}$$

$$\eta^{(2\,0)} = 0 \qquad \text{for the Dirac neutrino case,} \tag{2·35}$$

$$\eta^{(2\,0)} = (1 \pm 24)\cdot 10^{-3} \quad \text{for the Majorana neutrino case.} \tag{2·36}$$

These parameters are related to the weak mixing constants through the expressions Eqs. (2·29) – (2·32) in the lowest-order approximation. This is because the normalization factor $\mathcal{A}_{2\,0}$ takes the following form:

$$\mathcal{A}_{2\,0} = \left(1 + \lambda^2 + \kappa^2 + \eta^2\right) > 1 \qquad \text{for the Dirac neutrino case,} \tag{2·37}$$

$$\mathcal{A}_{2\,0} \approx \left[1 - \overline{u_e}^{\,2} - \overline{u_\mu}^{\,2} + \kappa^2 \left(\overline{v_\mu}^{\,2} + |\overline{w_{e\mu}}|^2\right) + \eta^2 \left(\overline{v_e}^{\,2} + |\overline{w_{e\mu\,h}}|^2\,\right)\right] \approx 1$$

for the Majorana neutrino case. (2·38)

We point out here that the spectrum $\mathcal{N}_{s\,t}(x)$ with $s \neq 0$ and/or $t \neq 0$ is derived from $\mathcal{N}_{0\,0}(x)$ by rearranging the standard function $n_1(x)$. Therefore, they should contain the same information on the weak mixing constants. Indeed, this situation is expressed formally by the following identity:

$$\mathcal{A}_{s\,t}\,\mathcal{N}_{s\,t}(x) = \mathcal{A}_{0\,0}\,\mathcal{N}_{0\,0}(x), \tag{2·39}$$

as seen from Eqs. (2·23) and (2·24). However, the normalization factors $\mathcal{A}_{s\,t}$ and $\mathcal{A}_{0\,0}$ themselves do not appear in the data analysis, and furthermore, the deviation functions are different in the spectrum functions $\mathcal{N}_{s\,t}(x)$ and $\mathcal{N}_{0\,0}(x)$. In §3, we examine the problem of obtaining the information mentioned above in the case that the experimental data are analyzed using different spectrum functions. For this purpose, the notation $\mathcal{N}_{s\,t}(x)$ is used to represent the spectrum with $s \neq 0$ and/or $t \neq 0$ hereafter.

There are some useful relations among the parameters in $\mathcal{N}_{0\,0}(x)$ and those in $\mathcal{N}_{s\,t}(x)$. The following two are obtained directly from the definitions in Eqs. (2·23) and (2·25):

$$\rho^{(0\,0)} = \frac{1}{\left(1 - s\,\rho^{(s\,t)} - t\,\eta^{(s\,t)}\right)}\rho^{(s\,t)}, \tag{2·40}$$

$$\eta^{(0\,0)} = \frac{1}{\left(1 - s\,\rho^{(s\,t)} - t\,\eta^{(s\,t)}\right)}\eta^{(s\,t)}. \tag{2·41}$$

Using these relations, numerical values of $\rho^{(0\,0)}$ and $\eta^{(0\,0)}$ can be estimated from experimental results for $\rho^{(2\,0)}$ and $\eta^{(2\,0)}$. We also have the identity*)

$$\left(1 + s\,\rho^{(0\,0)} + t\,\eta^{(0\,0)}\right)\left(1 - s\,\rho^{(s\,t)} - t\,\eta^{(s\,t)}\right) = 1, \tag{2·42}$$

from which we can derive the inverse relations to express $\rho^{(s\,t)}$ and $\eta^{(s\,t)}$ in terms of $\rho^{(0\,0)}$ and $\eta^{(0\,0)}$.

It can be shown by using these identities that the relation in Eq. (2·39) can be expressed as follows:

$$\mathcal{N}_{s\,t}(x) = \frac{1}{\left(1 + s\,\rho^{(0\,0)} + t\,\eta^{(0\,0)}\right)}\mathcal{N}_{0\,0}(x). \tag{2·43}$$

This implies that the spectrum function $\mathcal{N}_{s\,t}(x)$ and the parameters $(\rho^{(s\,t)}, \eta^{(s\,t)})$ can be obtained from knowledge about $\mathcal{N}_{0\,0}(x)$ and $(\rho^{(0\,0)}, \eta^{(0\,0)})$, and vice versa.

*) We can derive the following relations from the definitions given in Eqs. (2·23) and (2·25):

$$\mathcal{A}_{s\,t} = \left(1 + s\,\rho^{(0\,0)} + t\,\eta^{(0\,0)}\right)\mathcal{A}_{0\,0} \quad \text{or} \quad \mathcal{A}_{0\,0} = \left(1 - s\,\rho^{(s\,t)} - t\,\eta^{(s\,t)}\right)\mathcal{A}_{s\,t}.$$

2.2. *Anisotropic part of the spectrum: $\mathcal{P}(x)$*

Next, let us consider the anisotropic part, $\mathcal{P}(x)$, in Eq. (2·7). Here, we take the standard function $p_1(x)$ as the basis of the analysis. In order to elucidate the relation to the Michel parameterization, we define a common factor $\mathcal{B}_u$ by using the coefficient of $p_2(x)$ as

$$\mathcal{B}_u = a_- + u\,(k_{-\,c} + \varepsilon_m k_{-\,m}), \tag{2·44}$$

where u is some arbitrary number.*) Then, the anisotropic spectrum is written as $\mathcal{P}_{s\,t\,u}(x)$:

$$\mathcal{P}_{s\,t\,u}(x) = \xi^{(s\,t\,u)}\left\{p_1(x) + \left[p_2(x) - u\,p_1(x)\right]\delta^{(u)}\right\}, \tag{2·45}$$

where the parameters are defined as

$$\xi^{(s\,t\,u)} = \frac{\mathcal{B}_u}{\mathcal{A}_{s\,t}}, \qquad \delta^{(u)} = \frac{k_{-\,c} + \varepsilon_m\, k_{-\,m}}{\mathcal{B}_u}. \tag{2·46}$$

For the simple choice $(s,\, t,\, u) = (0,\, 0,\, 0)$, we have $\mathcal{A}_{0\,0} = a_+$ and $\mathcal{B}_0 = a_-$. Then, the anisotropic spectrum is expressed as

$$\mathcal{P}_{0\,0\,0}(x) = \xi^{(0\,0\,0)}\left[p_1(x) + p_2(x)\,\delta^{(0)}\right]. \tag{2·47}$$

This $\mathcal{P}_{0\,0\,0}(x)$ is identical to Eq. (2·7) if we set $\xi^{(0\,0\,0)} = (a_-/a_+)$.

The parameters $\xi^{(0\,0\,0)}$ and $\delta^{(0)}$ are related to the weak mixing constants. For the Dirac neutrino case, they are

$$\xi^{(0\,0\,0)} = \frac{(1-\lambda^2)}{(1+\lambda^2)}, \qquad \delta^{(0)} \approx \frac{1}{2}\left(\kappa^2 - \eta^2\right), \tag{2·48}$$

while for the Majorana neutrino case, they are

$$\xi^{(0\,0\,0)} \approx 1 - 2\lambda^2\overline{v_e}^{\,2}\overline{v_\mu}^{\,2} \approx 1, \quad \delta^{(0)} \approx \frac{1}{2}\left[\kappa^2\left(\overline{v_\mu}^{\,2} + |\overline{w_{e\mu}}|^2\right) - \eta^2\left(\overline{v_e}^{\,2} + |\overline{w_{e\mu\,h}}|^2\right)\right]. \tag{2·49}$$

Here, only the leading terms are given for the deviation from the standard model.

The Michel parameterization in Eq. (1·3) can be reproduced from $\mathcal{P}_{s\,t\,u}(x)$ by choosing $(s,\, t,\, u) = (2,\, 0,\, 6)$. In that case, the following correspondences are obtained:

$$p_2(x) - 6p_1(x) \simeq 6x^2(3-4x), \quad \xi^{(2\,0\,6)} = \xi_M, \quad \delta^{(6)} = \frac{2}{9}\left(\frac{3}{4} - \delta_M\right). \tag{2·50}$$

The experimental results reported by the Particle Data Group[2)] are as follows:

$$\left|\xi^{(2\,0\,6)}\,P_\mu\right| = 1.0027 \pm 0.0079 \pm 0.0030, \tag{2·51}$$

$$\delta^{(6)} = (1.1 \pm 2.7)\cdot 10^{-4}. \tag{2·52}$$

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*) In I, a similar common factor is denoted by $B_{n\,\ell}$. The presentation there can be corrected by taking $B_{n\,\ell} \to B_{n\,n}$ and $\delta_m \to 0$. Note that $B_{n\,n}$ there corresponds to $\mathcal{B}_{2n}$ in the present paper.

Here, P_μ represents the longitudinal polarization of the muon which is introduced in Eq. (1·1). The common parameter $\xi^{(2\,0\,6)}$ is related to the weak mixing constants as follows:

$$\xi^{(2\,0\,6)} = \frac{1-\lambda^2+3\,(\kappa^2-\eta^2)}{1+\lambda^2+\kappa^2+\eta^2} \qquad \text{for the Dirac neutrino case,} \tag{2·53}$$

$$\xi^{(2\,0\,6)} \approx 1+2\kappa^2(\overline{v_\mu}^2+\overline{w_{e\mu}}^2)-4\eta^2(\overline{v_e}^2+\overline{w_{e\mu h}}^2)\approx 1$$
$$\text{for the Majorana neutrino case.} \tag{2·54}$$

The parameter $\delta^{(6)}$ can be expressed in the same form as $\delta^{(0)}$ in Eqs. (2·48) and (2·49) if only the leading terms are kept for the deviation from the standard model.

The parameters in $\mathcal{P}_{s\,t\,u}(x)$ and $\mathcal{P}_{0\,0\,0}(x)$ satisfy the following identities:

$$\xi^{(0\,0\,0)} = \frac{\left[1-u\delta^{(u)}\right]}{\left[1-s\,\rho^{(s\,t)}-t\,\eta^{(s\,t)}\right]}\xi^{(s\,t\,u)}, \tag{2·55}$$

$$\delta^{(0)} = \frac{1}{\left[1-u\,\delta^{(u)}\right]}\,\delta^{(u)}. \tag{2·56}$$

The inverse relations are obtained by using the following identity.

$$\left(1+u\,\delta^{(0)}\right)\left(1-u\,\delta^{(u)}\right)=1. \tag{2·57}$$

All these relations can be derived from the definitions in Eqs. (2·44) and (2·46). Also, we can confirm the relation

$$\mathcal{P}_{s\,t\,u}(x) = \frac{1}{\left(1+s\,\rho^{(0\,0)}+t\,\eta^{(0\,0)}\right)}\,\mathcal{P}_{0\,0\,0}(x). \tag{2·58}$$

Note that this relation is independent of u introduced in Eq. (2·44).

Finally, it is useful to note that we have $\kappa=\eta$ if the $SU(2)_L\times SU(2)_R\times U(1)$ model is used, and in that case, the $\delta^{(u)}$ parameter becomes simpler:

$$\delta^{(u)} = 0 \qquad \text{for the Dirac neutrino case,} \tag{2·59}$$

$$\delta^{(u)} \simeq \frac{\eta^2}{2}\left[\left(\overline{v_\mu}^2-\overline{v_e}^2\right)+\left(|\,\overline{w_{e\mu}}\,|^2-|\,\overline{w_{e\mu\,h}}\,|^2\right)\right] \ll 1$$
$$\text{for the Majorana neutrino case.} \tag{2·60}$$

§3. Method of least squares

Here we find a sufficient condition for obtaining the same results for the weak mixing constants when different spectrum functions are adopted in the data analysis. Also, the method is formulated so as to include the QED radiative corrections appropriately. We first concentrate on the isotropic part, $\mathcal{N}_{s\,t}(x)$. It is easy to extend our treatment to the full spectrum in which the anisotropic part $\mathcal{P}_{s\,t\,u}(x)$ is taken into consideration.

3.1. *Analysis of the isotropic part of the spectrum: $\mathcal{N}(x)$*

We use the method of least squares in the data analysis. The QED radiative correction is not included for the time being. In the case that $\mathcal{N}_{s\,t}(x)$ is used, the unknown parameters ($\rho^{(s\,t)}$ and $\eta^{(s\,t)}$) are determined as those values at which the following $\chi^2_{s\,t}$ is minimized:

$$\chi^2_{s\,t} = \sum_i \frac{1}{\sigma_i^{\,2}} \Big[\mathcal{E}(x_i) - c_{s\,t}\,\mathcal{N}_{s\,t}(x_i)\Big]^2. \tag{3·1}$$

The summation over i runs over all measuring points x_i. The quantity $\mathcal{E}(x_i)$ represents an experimental datum at x_i, and σ_i is its experimental error. The global normalization constant $c_{s\,t}$ is introduced to adjust the theoretical values to the experimental data, so that the minimum point of $\chi^2_{s\,t}$ is sought under the variation of $c_{s\,t}$ as well as the parameters.

By requiring that $\chi^2_{s\,t}$ be a minimum, a set of analytical solutions is obtained for the parameters ($c_{s\,t}$, $c_{s\,t}\,\rho^{(s\,t)}$ and $c_{s\,t}\,\eta^{(s\,t)}$). They are known as the Cramers formula for a system of linear equations,[8] because these parameters appear linearly in $c_{s\,t}\mathcal{N}_{s\,t}(x)$. If the spectrum function $\mathcal{N}_{0\,0}(x)$ is adopted and $\chi^2_{0\,0}$ is required to have a minimum, the parameters ($c_{0\,0}$, $c_{0\,0}\,\rho^{(0\,0)}$ and $c_{0\,0}\,\eta^{(0\,0)}$) are expressed in terms of another set of analytical solutions. These two sets of solutions indicate that the two global normalization constants $c_{s\,t}$ and $c_{0\,0}$ satisfy the relation

$$\left(\frac{c_{s\,t}}{c_{0\,0}}\right) = \left(1 + s\,\rho^{(0\,0)} + t\,\eta^{(0\,0)}\right). \tag{3·2}$$

It can be confirmed further that, with this relation, the solutions for ($\rho^{(s\,t)}$, $\rho^{(0\,0)}$) and ($\eta^{(s\,t)}$, $\eta^{(0\,0)}$) are consistent with the relations in Eqs. (2·40) and (2·41), respectively. For this reason, the relation in Eq. (3·2) is called the "equivalent condition" hereafter.

If we combine Eq. (3·2) with Eq. (2·43), the following equality is obtained:

$$c_{s\,t}\,\mathcal{N}_{s\,t}(x) = c_{0\,0}\,\mathcal{N}_{0\,0}(x). \tag{3·3}$$

This implies that the χ^2-values are the same for different spectrum functions $\mathcal{N}_{s\,t}(x)$ and $\mathcal{N}_{0\,0}(x)$:

$$\chi^2_{s\,t} = \chi^2_{0\,0}. \tag{3·4}$$

In summary, the values of the parameters ($\rho^{(0\,0)}$ and $\eta^{(0\,0)}$) are determined experimentally, as those at which the $\chi^2_{0\,0}$-value is a minimum. The consistency of independent data analyses using $\mathcal{N}_{0\,0}(x)$ and $\mathcal{N}_{s\,t}(x)$ is guaranteed by the equivalent condition for the global normalization constants ($c_{0\,0}$ and $c_{s\,t}$). Then, $\chi^2_{s\,t}$ is equal to $\chi^2_{0\,0}$ when their parameters satisfy the relations in Eqs. (2·40) and (2·41). However, we note that the equivalent condition depends on a delicate balance of the global normalization constants, $c_{0\,0}$ and $c_{s\,t}$, because their difference is very slight, due to the smallness of $\rho^{(0\,0)}$ and $\eta^{(0\,0)}$.

Now let us examine a method for taking account of the QED radiative correction in the analysis of the spectrum. As a first step, we consider the case in which the data analysis is performed by assuming the standard model. Then, we attempt to find the minimum value of the quantity

$$X^2_{\rm sm} = \sum_i \frac{1}{{\sigma_i}^2} \Big| \mathcal{E}(x_i) - c_{\rm sm} \left[n_1(x_i) + f(x_i) \right] \Big|^2, \tag{3·5}$$

where $f(x)$ stands for the QED radiative correction associated with the standard function $n_1(x)$ in Eq. (2·8).[9),*)] Note that the only unknown parameter in $X^2_{\rm sm}$ is the global normalization constant $c_{\rm sm}$.

If the effect due to the coupling constants (λ, η and κ) is considered, then the above $X^2_{\rm sm}$ is modified. In that case, the standard function $n_1(x)$ is replaced by the spectrum function $\mathcal{N}_{0\,0}(x)$ or $\mathcal{N}_{s\,t}(x)$ in Eq. (2·24), but the QED radiative correction $f(x)$ is unchanged because of the consistency of the approximation. Thus, it is appropriate to introduce the following form instead of $X^2_{\rm sm}$:

$$X^2_{s\,t} = \sum_i \frac{1}{{\sigma_i}^2} \left[\mathcal{E}(x_i) - c_{s\,t} \mathcal{N}_{s\,t}(x_i) - c_{\scriptscriptstyle R}\, f(x_i) \right]^2. \tag{3·6}$$

Here, it is understood that the notation $(s,\, t)$ includes the case $(s\, =\, 0,\, t\, =\, 0)$. The two new parameters $c_{s\,t}$ and $c_{\scriptscriptstyle R}$ are a little different from $c_{\rm sm}$ in $X^2_{\rm sm}$, because $\mathcal{N}_{s\,t}(x)$ appears in place of $n_1(x)$. Note that this $c_{\scriptscriptstyle R}$ is independent of s and t. This can be confirmed by comparing two sets of analytical solutions for the parameters $(c_{0\,0},\, \rho^{(0\,0)},\, \eta^{(0\,0)},\, c_{\scriptscriptstyle R})$ and $(c_{s\,t},\, \rho^{(s\,t)},\, \eta^{(s\,t)},\, c_{\scriptscriptstyle R})$. These sets are obtained by requiring that $X^2_{0\,0}$ and $X^2_{s\,t}$, respectively, are minimal.

The equivalent condition in Eq. (3·2) is derived again for the new global normalization constants, $c_{0\,0}$ and $c_{s\,t}$, introduced in Eq. (3·6). Then, it can be proved under this equivalent condition that the χ^2-values are the same for different spectrum functions $\mathcal{N}_{0\,0}(x)$ and $\mathcal{N}_{s\,t}(x)$ whose parameters satisfy the relations in Eq. (2·40) for $(\rho^{(0\,0)},\, \rho^{(s\,t)})$ and the relations in Eq. (2·41) for $(\eta^{(0\,0)},\, \eta^{(s\,t)})$:

$$X^2_{s\,t} = X^2_{0\,0}. \tag{3·7}$$

We now give three comments. First, we examine whether the special choice of the spectrum function is preferable in the actual numerical analysis. For this purpose, it is useful to estimate the x dependence of the function $\varDelta(x)$ defined by

$$\varDelta(x) = \mathcal{E}(x) - c_{\rm sm} \left[n_1(x) + f(x) \right], \tag{3·8}$$

where $c_{\rm sm}$ is fixed such that $X^2_{\rm sm}$ in Eq. (3·5) is minimal. If this $\varDelta(x)$ possesses a clear x dependence, then we may choose such a value of s that $\varDelta(x)$ is roughly proportional to the deviation function $[n_2(x) - s\, n_1(x)]$ in Eq. (2·24). However, it is conceivable that $\varDelta(x)$ does not possess a clear x dependence, because of experimental errors. If this is the case, then it may be preferable to adopt $\mathcal{N}_{0\,0}(x)$ in Eq. (2·26),

*) The relation between our $f(x)$ and $F(x)$ of Arbuzov[9)] is

$$f(x) = \Big[1 + (m_e/m_\mu)^2 \Big]^{-4} F(x) - \mathcal{N}_{2\,0}(x).$$

because in that case, the parameters are related to the weak mixing constants in simpler forms. In conclusion, we propose to use $X^2_{0\,0}$ in the actual data analysis because of its simplicity.

The next comment is that the precise determination of the value of $X^2_{\rm sm}$ itself, defined in Eq. (3·5), is interesting. This is because a large deviation from the standard model is not expected, especially in the Majorana neutrino case. This is discussed in the last paragraph of §4.

The final comment is that, in contrast to $X^2_{s\,t}$ in Eq. (3·6), the definition

$$Y^2_{s\,t} = \sum_i \frac{1}{\sigma_i^2} \Big| \mathcal{E}(x_i) - c_{s\,t} \left[\mathcal{N}_{s\,t}(x_i) + f(x_i) \right] \Big|^2 \tag{3·9}$$

is inappropriate theoretically, because it leads to the inequality $Y^2_{s\,t} \neq Y^2_{0\,0}$.

3.2. *Analysis of the full spectrum: $\mathcal{D}(x)$*

In the extended form of the parameterization, the full spectrum appearing in Eq. (2·4) is expressed as

$$\mathcal{D}_{s\,t\,u}(x,\,\theta) = \Big[\mathcal{N}_{s\,t}(x) + P_\mu \cos\theta\, \mathcal{P}_{s\,t\,u}(x) \Big], \tag{3·10}$$

where $\mathcal{N}_{s\,t}(x)$ and $\mathcal{P}_{s\,t\,u}(x)$ are given by Eqs. (2·24) and (2·45), respectively. The method of least squares can be applied to $\mathcal{D}_{s\,t\,u}(x,\,\theta)$ similarly to the isotropic part, $\mathcal{N}_{s\,t}(x)$.

We now summarize the essential points for the case with no radiative correction. The new $\chi^2_{s\,t\,u}$ is defined as

$$\chi^2_{s\,t\,u} = \sum_{i,\,j} \frac{1}{\sigma^2_{i\,j}} \Big[\mathcal{E}(x_i,\,\theta_j) - d_{s\,t}\,\mathcal{D}_{s\,t\,u}(x_i,\,\theta_j) \Big]^2, \tag{3·11}$$

where x_i and θ_j are sets of observed quantities at one measuring point. We can obtain analytical solutions for two new parameters ($\xi^{(s\,t\,u)}$ and $\delta^{(u)}$), in addition to three old ones ($d_{s\,t}$, $\rho^{(s\,t)}$ and $\eta^{(s\,t)}$), by requiring that $\chi^2_{s\,t\,u}$ be minimal. The corresponding solutions are also obtained by treating $\chi^2_{0\,0\,0}$. It should be noted that the global normalization constant $d_{s\,t}$ here depends only on s and t, because it is determined as a coefficient for $\mathcal{N}_{s\,t}(x)$ of $\mathcal{D}_{s\,t\,u}(x,\,\theta)$ in the method of least squares. Then, it can be verified not only that the global normalization constants satisfy an equivalent condition similar to that in Eq. (3·2) but also that the other four parameters are consistent with the relations in Eqs. (2·40), (2·41), (2·55) and (2·56). Due to this equivalent condition, together with the relations in Eqs. (2·43) and (2·58), we have the identity $d_{s\,t}\mathcal{D}_{s\,t\,u}(x,\,\theta) = d_{0\,0}\mathcal{D}_{0\,0\,0}(x,\,\theta)$, and hence the equality

$$\chi^2_{s\,t\,u} = \chi^2_{0\,0\,0}. \tag{3·12}$$

In the case that the radiative QED effect is taken into consideration, we modify $\chi^2_{s\,t\,u}$ and define the following $Z^2_{s\,t\,u}$, which satisfies both the consistency conditions for parameters and the equality $Z^2_{s\,t\,u} = Z^2_{0\,0\,0}$:

$$Z^2_{s\,t\,u} = \sum_{i,\,j} \frac{1}{\sigma^2_{i\,j}} \Big[\mathcal{E}(x_i,\,\theta_j) - d_{s\,t}\,\mathcal{D}_{s\,t\,u}(x_i,\,\theta_j) - d_R\,F(x_i,\,\theta_j) \Big]^2, \tag{3·13}$$

where the new parameter d_R corresponds to c_R in Eq. (3·6) and

$$F(x,\,\theta) = \big[f(x) + P_\mu\,\cos\theta\, g(x)\,\big]. \tag{3·14}$$

Here, $g(x)$ represents the QED radiative correction associated with the anisotropic standard function $p_1(x)$ in Eq. (2·11).[9),*)]

Corresponding to $X^2_{\rm sm}$ for the isotropic part of the standard model, the following $Z^2_{\rm sm}$ with the global normalization constant $d_{\rm sm}$ is defined for the full spectrum:

$$Z^2_{\rm sm} = \sum_{i,\,j} \frac{1}{\sigma^{\,2}_{i\,j}} \Big|\mathcal{E}(x_i,\,\theta_j) - d_{\rm sm}\,\left[D_{\rm sm}(x_i,\,\theta_j) + F(x_i,\,\theta_j)\right]\Big|^2, \tag{3·15}$$

where

$$D_{\rm sm}(x,\,\theta) = \big[n_1(x) + P_\mu\,\cos\theta\,p_1(x)\,\big]. \tag{3·16}$$

It is interesting to compare the minima of $Z^2_{\rm sm}$ and $Z^2_{0\,0\,0}$ in order to directly determine the extent of the departure from the standard model.

§4. Discussion

Let us consider a possible method for determining whether a neutrino is of Dirac or Majorana type. This is facilitated by observing the $\eta^{(s\,t)}$ parameter, which is zero or nonzero, depending on the Dirac or Majorana neutrino within the frame of the gauge theory, as seen from Eqs. (2·30) and (2·32).[**)] The Michel parameter η_M has been widely used as a measure to quantify the deviation from the standard model. It corresponds to $\eta^{(2\,0)}$ phenomenologically, as mentioned in the paragraph including Eq. (2·36). However, the $\eta^{(s\,t)}$ term is defined for the Majorana neutrino case within the framework of gauge theory, while the η_M term comes from the interference between the $(V \pm A)$ and $(S \pm P)$ (or T) forms, even in the massless Dirac neutrino case.[2)] In any case, determination of this η parameter reveals the deviation from the standard model.

It is well known that measuring the η parameter is very difficult experimentally. One reason is that the contribution of the relevant deviation function, $n_3(x)$, is small, because it is proportional to the small quantity through x_0, as shown in Eq. (2·13). We can avoid this problem by considering the τ-decay

$$\tau^+ \to \mu^+ + \nu_\mu + \overline{\nu_\tau}. \tag{4·1}$$

All formula in the previous sections can be applied to this τ-decay through the replacement of both $(m_\mu \to m_\tau)$ and $(m_e \to m_\mu)$. Then, the value $x_0 = m_e/W \simeq$

*) The relation between our $g(x)$ and $G(x)$ of Arbuzov[9)] is

$$g(x) = -\Big[1 + (m_e/m_\mu)^2\Big]^{-4} G(x) - \mathcal{P}_{2\,0}(x).$$

**) This difference is independent of the choice of the normalization factor A. It should be noted that there is some misleading discussion in I regarding this point. In particular, it is not the case that there is some difference between the Dirac and Majorana cases by choosing A.

0.01 is shifted to $x_0(\tau) = m_\mu/W_\tau \simeq 0.12$, where $W_\tau = (m_\tau^2 + m_\mu^2)/2m_\tau$. The second reason for the difficulty in detecting the $\eta^{(s\,t)}$ parameter is its smallness within gauge theory, as seen from Eqs. (2·32) and (2·21). Its rough estimate was discussed in §4.2 of I. We conclude that muon decay cannot be used to discriminate between the Majorana and Dirac neutrino cases in actual experiments.

Next, let us examine the order of magnitude of the normalization factor A. In the conventional model used to analyze the experimental results, it is assumed that $A = \mathcal{A}_{2\,0} = 1$.[10] However, in our model for the Dirac neutrino, as we know $\mathcal{A}_{0\,0} = \left(1 + \lambda^2\right) > 1$ from Eq. (2·27), we need some information concerning λ^2. We may conclude that $\lambda^2 < O(10^{-3})$ from the data given in Eqs. (2·51) – (2·53) and (2·34). The information on λ^2 can be directly extracted from the experimental results on the longitudinal polarization of the emitted positron. We have the following expression for this polarization in the direction perpendicular to the muon spin polarization, namely $\theta = \pi/2$:

$$P_L(x,\, \theta = \pi/2) = \xi^{(0\,0\,0)}\, \frac{x_p\,[\, q_1(x) + q_2(x)\, \delta^{(0)}\,]}{[\, n_1(x) + n_2(x)\, \rho^{(0\,0)} + n_3(x)\, \eta^{(0\,0)}\,]}, \tag{4·2}$$

which is obtained from Eq. (A·3) in Appendix A. In the limit of the maximum of x, $x \to x_{\max} \simeq 1$, the ratio $[\, x_p\, q_1(x)/n_1(x)]$ approaches to $(x_p/x) \simeq 1$, while all others, $n_2(x)$, $n_3(x)$ and $q_2(x)$, vanish. Then, we have $P_L(x,\, \theta = \pi/2) \to \xi^{(0\,0\,0)} \approx (1 - 2\,\lambda^2)$, as seen from Eq. (2·48). Thus, the restriction $\lambda^2 < 2 \cdot 10^{-2}$ is obtained from the present average value on this polarization.[2] In the Majorana neutrino case, at present, there is no definite information, although it can be imagined from Eq. (2·28) that the deviation from $\mathcal{A}_{0\,0} = 1$ is very small.

Finally, we comment on the data for the Michel parameter ρ_M. The mean value of $\rho^{(2\,0)}$ obtained from ρ_M is $\rho^{(2\,0)} = -6 \cdot 10^{-4}$, as shown in Eq. (2·34). This mean value is negative, although it could be positive, within the experimental uncertainty. Theoretically, it is predicted to be positive in the case of the Hamiltonian given in Eq. (2·1), as seen from Eq. (2·29) or (2·31) and (2·40). There is a possibility that this difference results from some ambiguity in the data analysis. This is because the consistency of the data analysis depends delicately on the equivalent condition in Eq. (3·2) and the method of treating the QED radiative correction, as mentioned in §3. Under these circumstances, it is of interest to compare experimental results for $\rho^{(2\,0)}$ and $\rho^{(0\,0)}$, which can be determined by using $\mathcal{N}_{2\,0}(x)$ and $\mathcal{N}_{0\,0}(x)$, respectively. These parameters should satisfy the relation given in Eq. (2·40) and have the same signature. In connection to this, the evaluation of $X^2_{0\,0}$ and $X^2_{2\,0}$ is important, because, according to the theory, they are equal. It would also be interesting to compare them with $X^2_{\rm sm}$ in Eq. (3·5), because a large deviation from the standard model is not expected, especially for the Majorana neutrino case.

Appendix A
—— *Polarization of the Positron* ——

Because the parameters are defined here in somewhat different forms than in I, we list expressions for the polarization of an emitted positron.

The differential decay rate is expressed as

$$\frac{d^2\Gamma}{dx\,d\cos\theta} = \frac{1}{2}\Gamma_W\,A\,\mathcal{D}(x,\,\theta)\,\left[1+\vec{P}_e(x,\,\theta)\cdot\hat{\zeta}\right], \tag{A·1}$$

where the vector $\vec{P}_e(x,\,\theta)$ is the polarization vector of e^+, and $\hat{\zeta}$ is the directional vector of the measurement of the e^+ spin polarization. The decay plane is defined by the direction of the momentum ($\vec{p_e}$) of e^+ and the muon polarization vector ($\vec{P_\mu}$).

The three components of the e^+ spin polarization vector are defined as[2)]

$$\vec{P}_e(x,\,\theta) = P_L(x,\,\theta)\hat{p_e} + P_{T1}(x,\,\theta)\frac{(\hat{p_e}\times\vec{P_\mu})\times\hat{p_e}}{|(\hat{p_e}\times\vec{P_\mu})\times\hat{p_e}|} + P_{T2}(x,\,\theta)\frac{\hat{p_e}\times\vec{P_\mu}}{|\hat{p_e}\times\vec{P_\mu}|}. \tag{A·2}$$

Explicit expressions for $P_L(x,\,\theta)$, $P_{T1}(x,\,\theta)$ and $P_{T2}(x,\,\theta)$ are presented here in terms of the parameters defined in §2 of the present paper. For simplicity, they are listed only in the simple form with both the normalization factor $\mathcal{A}_{0\,0}$ and the common factor $\mathcal{B}_0$. Also, the radiative corrections are not included here.[9)]

A.1. *Longitudinal polarization:* $P_L(x,\,\theta)$

It is convenient to separate the isotropic and anisotropic distributions of e^+ with respect to the muon polarization vector $\vec{P_\mu}$, namely,

$$P_L(x,\,\theta) = \frac{Q(x) + P_\mu\,\cos\theta\,S(x)}{D(x,\,\theta)}, \tag{A·3}$$

where the denominator $D(x,\,\theta)$ is defined from Eq. (3·10) as follows:

$$D(x,\,\theta) = \frac{1}{x_p}\,\mathcal{D}_{0\,0\,0}(x,\,\theta) = \frac{1}{x_p}[\mathcal{N}_{0\,0}(x) + P_\mu\,\cos\theta\,\mathcal{P}_{0\,0\,0}(x)]. \tag{A·4}$$

The isotropic part, $Q(x)$, and anisotropic part, $S(x)$, of the longitudinal polarization are expressed as

$$Q(x) = \xi^{(0\,0\,0)}\,\left[q_1(x) + q_2(x)\,\delta^{(0)}\right], \tag{A·5}$$

$$S(x) = \left[s_1(x) + s_2(x)\,\rho^{(0\,0)} + s_3(x)\,\eta^{(0\,0)}\right], \tag{A·6}$$

where

$$q_1(x) = x_p\,(3 - 2\,x - r_0^2), \tag{A·7}$$

$$q_2(x) = 12\,x_p\,(1-x), \tag{A·8}$$

$$s_1(x) = (-x + 2\,x^2 - x_0^2), \tag{A·9}$$

$$s_2(x) = 12\,x(1-x), \tag{A·10}$$

$$s_3(x) = -2\,x_0\,(1-x). \tag{A·11}$$

The parameters ($\xi^{(0\,0\,0)}$ and $\delta^{(0)}$) in $Q(x)$ are defined in Eq. (2·46) for the case of $\mathcal{P}(x)$, while the parameters ($\rho^{(0\,0)}$ and $\eta^{(0\,0)}$) in $S(x)$ are defined in Eq. (2·25) for the case of $\mathcal{N}(x)$.

A.2. *Transverse polarization within the decay plane:* $P_{T1}(x, \theta)$

The x dependent part, $R(x)$, of $P_{T1}(x, \theta)$ is defined as

$$P_{T1}(x, \theta) = \frac{P_\mu \sin\theta\, R(x)}{D(x, \theta)}, \tag{A·12}$$

with

$$R(x) = \left[r_1(x)\,(1 - 12\,\rho^{(0\,0)}) + r_2(x)\,\eta^{(0\,0)} \right], \tag{A·13}$$

where

$$r_1(x) = -x_0\,(1 - x), \tag{A·14}$$
$$r_2(x) = -2\,(x - x_0^2). \tag{A·15}$$

Note that the small quantity x_0 appears in $r_1(x)$, which represents the prediction obtained from the standard model. The quantity, $\eta^{(0\,0)}$, which indicates the existence of the Majorana neutrino, is associated with the larger deviation function, $r_2(x)$.

A.3. *Transverse polarization perpendicular to the decay plane:* $P_{T2}(x, \theta)$

The x dependent part, $T(x)$, of $P_{T2}(x, \theta)$ is defined as

$$P_{T2}(x, \theta) = \frac{P_\mu \sin\theta\, T(x)}{D(x, \theta)}, \tag{A·16}$$

where

$$T(x) = 2\,x_p\left(1 - r_0^2\right)\,\eta_{im}^{(0\,0)}. \tag{A·17}$$

Here, the new parameter $\eta_{im}^{(0\,0)}$ is defined as follows:[3)]

$$\eta_{im}^{(0\,0)} = \varepsilon_m \left(\frac{\lambda}{\mathcal{A}^{(0\,0)}} \right) \mathrm{Im}(\overline{w_{e\mu}}^{\,*}\ \overline{w_{e\mu\,h}}). \tag{A·18}$$

A non-zero value of $T(x)$ implies the existence of a non-zero Majorana *CP* violating phase in our model. There is no corresponding term in either the standard model or our model for the Dirac neutrino.

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